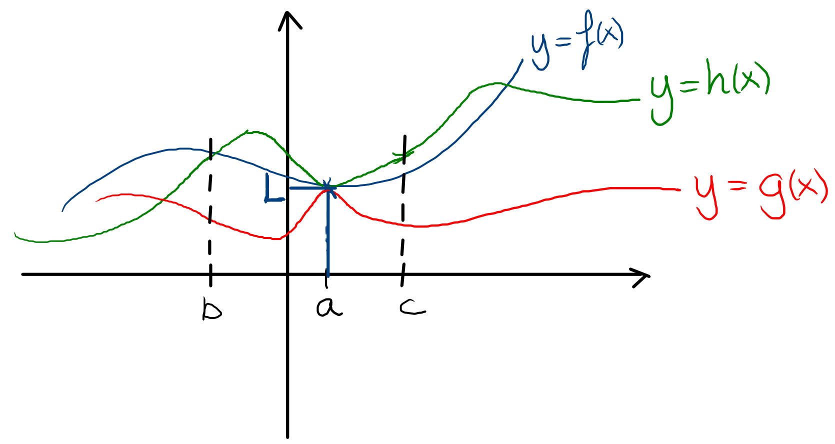
Théorème des deux gendarmes

Soit $h(x) \leqslant f(x) \leqslant g(x)$ pour $x \in]b;c[$ contenant a

Si $\lim\limits_{x \to a} h(x) = \lim\limits_{x \to a} g(x) = L$

Alors $\lim\limits_{x \to a} f(x) = L$



Exples

1) $\lim\limits_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = \text{“} 0 \cdot \sin(\infty) \text{”} \underset{\text{2 gendarmes}}{\overset{\text{thm. des}}{=}} 0$

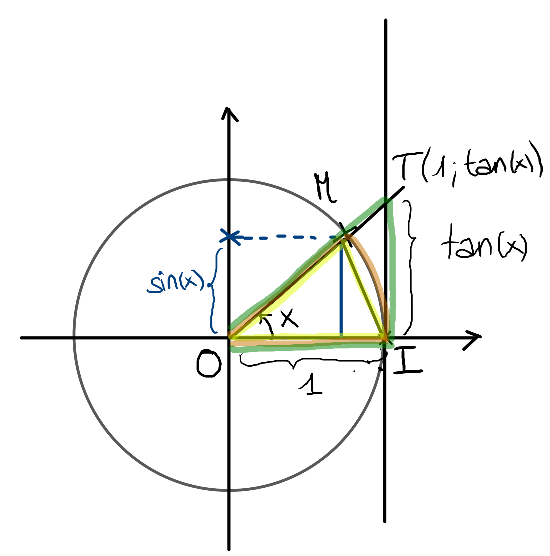
comme $-1 \leqslant \sin(x) \leqslant 1$

$-1 \leqslant \sin\left(\frac{1}{x}\right) \leqslant 1 \qquad \Big| \cdot x$

$-x \leqslant x \sin\left(\frac{1}{x}\right) \leqslant x$

$\underset{0}{\downarrow_{x \to 0}} \qquad \underset{0}{\downarrow_{x \to 0}} \qquad \underset{0}{\downarrow_{x \to 0}}$

2) $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} \;\; \text{“}\!=\!\frac{0}{0}\text{”} \;\; f.i.$



$\text{Aire}\left(\triangle OIM\right) = \dfrac{1 \cdot \sin(x)}{2} = \dfrac{\sin(x)}{2}$

$\text{Aire}\left(\text{sect. circ. } OIM\right) = \dfrac{\pi \cdot 1^2 \cdot x}{2\pi} = \dfrac{x}{2}$

$\text{Aire}\left(\triangle OIT\right) = \dfrac{1 \cdot \tan(x)}{2} = \dfrac{\tan(x)}{2}$

$\Rightarrow \quad \dfrac{\sin(x)}{2} \leqslant \dfrac{x}{2} \leqslant \dfrac{\tan(x)}{2}$

$\Leftrightarrow \quad \sin(x) \leqslant x \leqslant \tan(x) = \dfrac{\sin(x)}{\cos(x)}$

$\Leftrightarrow \quad \dfrac{1}{\sin(x)} \geqslant \dfrac{1}{x} \geqslant \dfrac{\cos(x)}{\sin(x)} \qquad \Big| \cdot \sin(x)$

$\Leftrightarrow \quad 1 \geqslant \dfrac{\sin(x)}{x} \geqslant \cos(x)$

$\underset{1}{\downarrow_{x \to 0}} \qquad \underset{1}{\downarrow_{x \to 0}} \qquad \underset{1}{\downarrow_{x \to 0}}$

$\Rightarrow \quad \boxed{\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = 1}$

Rappel : $\sin^2(x) + \cos^2(x) = 1$

3) $$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} \cdot \frac{1+\cos(x)}{1+\cos(x)} = \lim_{x\to 0} \frac{1-\cos^2(x)}{x^2(1+\cos(x))} = \lim_{x\to 0} \frac{\sin^2(x)}{x^2(1+\cos(x))}$$

$$= \lim_{x\to 0} \underbrace{\left(\frac{\sin(x)}{x}\right)^2}_{1^2} \cdot \underbrace{\frac{1}{1+\cos(x)}}_{\frac{1}{2}} = \frac{1}{2}$$

4) $$\lim_{x\to 0} \frac{\sin(2x)}{5x} = \lim_{x\to 0} \frac{\sin(2x)}{2x} \cdot \frac{2}{5} = \lim_{t\to 0} \frac{\sin(t)}{t} \cdot \frac{2}{5} = 1 \cdot \frac{2}{5} = \frac{2}{5}$$

chgmt de variable : $\underset{\downarrow}{t} = \underset{\downarrow}{2x}$
$\quad 0 \quad\quad 0$

Variante :

$$\lim_{x\to 0} \frac{\sin(2x)}{5x} \underset{\text{form.}}{=} \lim_{x\to 0} \frac{2\sin(x)\cos(x)}{5x} = \lim_{x\to 0} \underbrace{\frac{\sin(x)}{x}}_{1} \cdot \underbrace{\frac{2\cos(x)}{5}}_{\frac{2\cdot 1}{5}} = 1 \cdot \frac{2}{5} = \frac{2}{5}$$