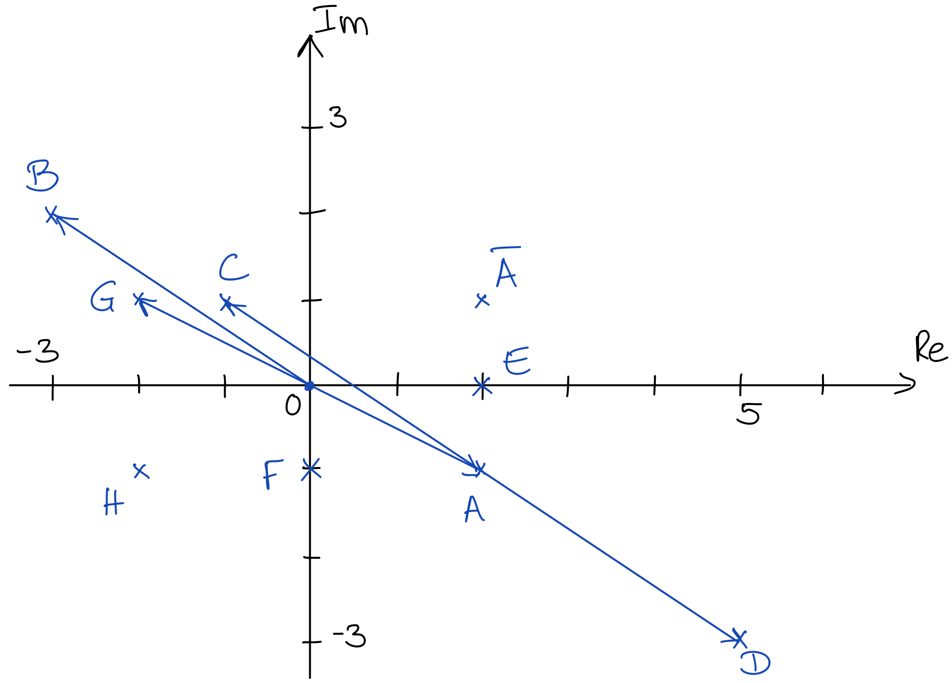
Ex 1.2.1



$\vec{OC} = \vec{OA} + \vec{OB}$ $\qquad$ $\vec{OD} = \vec{OA} - \vec{OB}$ $\qquad$ $\vec{OG} = -\vec{OA}$

Ex 1.2.2

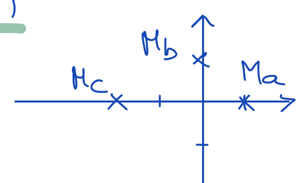
a) $1 = 1(\cos(0) + i\sin(0)) = \text{cis}(0)$

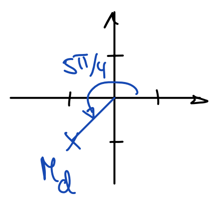
b) $i = \text{cis}\left(\frac{\pi}{2}\right)$

c) $-2 = 2\,\text{cis}(\pi)$

d) $-1-i = \sqrt{2}\,\text{cis}\left(\frac{5\pi}{4}\right)$

graphiquement





- $r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$
- $\tan(\theta) = \frac{b}{a} = \frac{-1}{-1} = 1 \Leftrightarrow \theta = \frac{\pi}{4} + k\cdot\pi \overset{\text{quadrant}}{\underset{\text{III}}{\Longrightarrow}} \theta = \frac{\pi}{4} + 1\cdot\pi = \frac{5\pi}{4}$

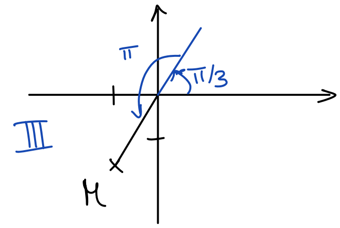
Variante : $\cos(\theta) = \frac{a}{r} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \Leftrightarrow \theta = \begin{cases} 3\pi/4 + k\cdot 2\pi \\ -3\pi/4 + k\cdot 2\pi \end{cases}$

$\overset{\text{quadrant}}{\underset{\text{III}}{\Longrightarrow}} \theta = -\frac{3\pi}{4} + 0\cdot 2\pi = -\frac{3\pi}{4}$ ou $\frac{5\pi}{4}$

Variante : $\sin(\theta) = \frac{b}{r} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \Leftrightarrow \theta = \begin{cases} -\pi/4 + k\cdot 2\pi \\ \underbrace{\pi - (-\pi/4)}_{5\pi/4} + k\cdot 2\pi \end{cases}$

$\overset{\text{quadrant}}{\underset{\text{III}}{\Longrightarrow}} \theta = \frac{5\pi}{4} + 0\cdot 2\pi = \frac{5\pi}{4}$

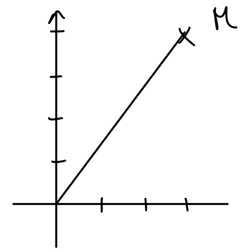
e) $-1-\sqrt{3}\,i = 2\,\text{cis}\left(\frac{4\pi}{3}\right)$



- $|z| = \sqrt{(-1)^2+(-\sqrt{3})^2} = \sqrt{1+3} = 2$
- $\tan(\theta) = \frac{b}{a} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$

$\Leftrightarrow \quad \theta = \frac{\pi}{3} + k\cdot\pi \quad \overset{III}{\Longrightarrow} \quad \theta = \frac{\pi}{3} + 1\cdot\pi = \frac{4\pi}{3}$

f) $3+4i \cong 5\,\text{cis}(0{,}927)$



- $|z| = \sqrt{9+16} = 5$
- $\tan(\theta) = \frac{b}{a} = \frac{4}{3} \quad \Leftrightarrow \quad \theta \cong 53{,}13° + 2\cdot 180°$

$\cong 0{,}927 + k\cdot\pi$

$\overset{I}{\Longrightarrow} \quad \theta \cong 53{,}13°$ ou $0{,}927$

Ex 1.2.3

a) $4\,\text{cis}\left(-\frac{\pi}{3}\right) = 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = 4\left(\frac{1}{2} - i\cdot\frac{\sqrt{3}}{2}\right) = 2 - 2\sqrt{3}\,i$

b) $\frac{3}{4}\,\text{cis}\left(\frac{3\pi}{4}\right) = \frac{3}{4}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = \frac{3}{4}\left(-\frac{\sqrt{2}}{2} + i\cdot\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{8} + \frac{3\sqrt{2}}{8}i$

c) $\pi\,\text{cis}(\pi) = \pi\left(\cos(\pi) + i\sin(\pi)\right) = \pi(-1 + i\cdot 0) = -\pi$

d) $4\,\text{cis}\left(\frac{\pi}{3}\right) = 4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = 4\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right) = 2 + 2\sqrt{3}\,i$

e) $\text{cis}\left(-\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = 0 + i\cdot(-1) = -i$

Ex 1.2.4

a) $2\,\text{cis}\left(\frac{\pi}{4}\right)\cdot 3\,\text{cis}\left(\frac{\pi}{6}\right) = 2\cdot 3\,\text{cis}\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = 6\,\text{cis}\left(\frac{5\pi}{6}\right)$

b) $6\,\text{cis}\left(\frac{2\pi}{3}\right) : 3\,\text{cis}\left(-\frac{\pi}{3}\right) = \frac{6}{3}\,\text{cis}\left(\frac{2\pi}{3} - \left(-\frac{\pi}{3}\right)\right) = 2\,\text{cis}(\pi)$

c) $\left(2\,\text{cis}\left(\frac{\pi}{3}\right)\right)^3 = 2\cdot 2\cdot 2\,\text{cis}\left(\frac{\pi}{3}+\frac{\pi}{3}+\frac{\pi}{3}\right) = 2^3\,\text{cis}\left(3\cdot\frac{\pi}{3}\right) = 8\,\text{cis}(\pi)$

Ex 1.2.5

a) $z_1 = -1+i = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$ car $r_1 = |z_1| = \sqrt{2}$, quadrant II, $\tan(\theta) = \frac{1}{-1} = -1 \Rightarrow \theta = \frac{3\pi}{4}$

$z_2 = 1+i = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$ car $r_2 = |z_2| = \sqrt{2}$, " I, $\tan(\theta) = \frac{1}{1} = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$z_1 \cdot z_2 = \sqrt{2}\cdot\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}+\frac{\pi}{4}\right) = \underline{2\,\text{cis}(\pi)} = \underline{-2}$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{\sqrt{2}}\,\text{cis}\left(\frac{3\pi}{4}-\frac{\pi}{4}\right) = \underline{\text{cis}\left(\frac{\pi}{2}\right)} = \underline{i}$$

b) $z_1 = -2-2\sqrt{3}i = 4\,\text{cis}\left(\frac{4\pi}{3}\right)$

car $r_1 = \sqrt{4+4\cdot 3} = 4$, quadrant III, $\tan(\theta) = \frac{-2\sqrt{3}}{-2} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}+\pi = \frac{4\pi}{3}$

$z_2 = 5i = 5\,\text{cis}\left(\frac{\pi}{2}\right)$

$$z_1 \cdot z_2 = 4\cdot 5\,\text{cis}\left(\frac{4\pi}{3}+\frac{\pi}{2}\right) = \underline{20\,\text{cis}\left(\frac{11\pi}{6}\right)} = 20\left(\frac{\sqrt{3}}{2}+i\cdot\left(-\frac{1}{2}\right)\right) = \underline{10\sqrt{3}-10i}$$

$$\frac{z_1}{z_2} = \frac{4}{5}\,\text{cis}\left(\frac{4\pi}{3}-\frac{\pi}{2}\right) = \underline{\frac{4}{5}\,\text{cis}\left(\frac{5\pi}{6}\right)} = \frac{4}{5}\left(-\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}\right) = \underline{-\frac{2\sqrt{3}}{2}+\frac{2}{5}i}$$

c) $z_1 = 2i = 2\,\text{cis}\left(\frac{\pi}{2}\right)$

$z_2 = -3i = 3\,\text{cis}\left(\frac{3\pi}{2}\right)$

$$z_1 \cdot z_2 = 2\cdot 3\,\text{cis}\left(\frac{\pi}{2}+\frac{3\pi}{2}\right) = 6\,\text{cis}(2\pi) = \underline{6}$$

$$\frac{z_1}{z_2} = \frac{2}{3}\,\text{cis}\left(\frac{\pi}{2}-\frac{3\pi}{2}\right) = \frac{2}{3}\,\text{cis}(-\pi) = \underline{-\frac{2}{3}}$$

d) $z_1 = -10 = 10\,\text{cis}(\pi)$

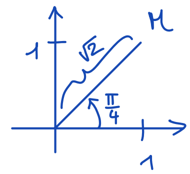
$z_2 = -4 = 4\,\text{cis}(\pi)$

$$z_1 \cdot z_2 = 10\cdot 4\,\text{cis}(\pi+\pi) = 40\,\text{cis}(2\pi) = \underline{40}$$

$$\frac{z_1}{z_2} = \frac{10}{4}\,\text{cis}(\pi-\pi) = \frac{5}{2}\,\text{cis}(0) = \underline{\frac{5}{2}}$$

Ex 1.2.8

a) $z^4 = 1+i = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$



Soit $z = t\,\text{cis}(\alpha)$

$\Rightarrow \quad t^4\,\text{cis}(4\alpha) = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

$\Rightarrow \begin{cases} t^4 = \sqrt{2} \\ 4\alpha = \frac{\pi}{4} + k\cdot 2\pi \end{cases} \Leftrightarrow \begin{cases} t = \sqrt[8]{2} \\ \alpha = \left(\frac{\pi}{4} + k\cdot 2\pi\right)\cdot\frac{1}{4} = \frac{\pi}{16} + k\cdot\frac{\pi}{2} \end{cases}, \quad k = 0, 1, \dots, 3$

$\Rightarrow \quad z_0 = \sqrt[8]{2}\,\text{cis}\left(\frac{\pi}{16}\right)$

$z_1 = \sqrt[8]{2}\,\text{cis}\left(\frac{\pi}{16} + \frac{\pi}{2}\right) = \sqrt[8]{2}\,\text{cis}\left(\frac{9\pi}{16}\right)$

$z_2 = \sqrt[8]{2}\,\text{cis}\left(\frac{\pi}{16} + 2\cdot\frac{\pi}{2}\right) = \sqrt[8]{2}\left(\frac{17\pi}{16}\right)$

$z_3 = \sqrt[8]{2}\,\text{cis}\left(\frac{\pi}{16} + 3\cdot\frac{\pi}{2}\right) = \sqrt[8]{2}\,\text{cis}\left(\frac{25\pi}{16}\right)$

b) $z^4 = 24i - 7$

posons $z^2 = w = a+bi \Leftrightarrow z^4 = w^2 = (a+bi)^2 = 24i - 7$

$\Leftrightarrow \quad a^2 - b^2 + 2abi = 24i - 7$

$\Rightarrow \begin{cases} a^2+b^2 = \sqrt{24^2+7^2} = 25 \\ a^2-b^2 = -7 \\ 2ab = 24 \end{cases} \begin{vmatrix} 1 \\ 1 \end{vmatrix} \begin{matrix} 1 \\ -1 \end{matrix} \Rightarrow \begin{cases} 2a^2 = 18 \\ 2b^2 = 32 \\ ab = 12 \end{cases} \Rightarrow \begin{cases} a = \pm 3 \\ b = \pm 4 \end{cases}$

($ab = 12$: a et b de m̂ signe)

$\Rightarrow \quad z_1^2 = 3+4i \quad$ et $\quad z_2^2 = -3-4i$

posons $z_{1,2} = c_{1,2} + d_{1,2}i$

$\Rightarrow \quad (c_1+d_1i)^2 = 3+4i \quad$ et $\quad (c_2+d_2i)^2 = -3-4i$

$c_1^2 - d_1^2 + 2c_1d_1i = 3+4i$

$\Rightarrow \begin{cases} c_1^2+d_1^2 = 5 \\ c_1^2-d_1^2 = 3 \\ 2c_1d_1 = 4 \end{cases} \begin{vmatrix} 1 \\ 1 \end{vmatrix} \begin{matrix} 1 \\ -1 \end{matrix}$

$\Leftrightarrow \begin{cases} 2c_1^2 = 8 \\ 2d_1^2 = 2 \\ c_1d_1 = 2 \end{cases} \Leftrightarrow \begin{cases} c_1 = \pm 2 \\ d_1 = \pm 1 \end{cases}$

$\Rightarrow \quad z_{11} = 2+i \quad$ et $\quad z_{12} \; -2-i$

$\Rightarrow \begin{cases} c_2^2+d_2^2 = 5 \\ c_2^2-d_2^2 = -3 \\ 2c_2d_2 = -4 \end{cases} \begin{vmatrix} 1 \\ 1 \end{vmatrix} \begin{matrix} 1 \\ -1 \end{matrix}$

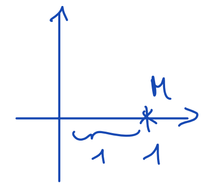
$\Leftrightarrow \begin{cases} 2c_2^2 = 2 \\ 2d_2^2 = 8 \\ c_2d_2 = -2 \end{cases} \Leftrightarrow \begin{cases} c_2 = \pm 1 \\ d_2 = \mp 2 \end{cases}$

($c_2d_2 = -2$: de signe contraire)

$\Rightarrow \quad z_{21} = 1-2i \quad$ et $\quad z_{22} = -1+2i$

Ex 1.2.9

a) racines 7e de l'unité   on cherche w tq $w^7 = 1$



on écrit 1 sous sa forme trigo : $1 = cis(0)$

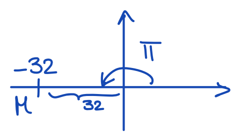
on pose $w = t\,cis(\alpha)$ $\Rightarrow$ $w^7 = t^7\,cis(7\alpha) = cis(0)$

$$\Rightarrow \begin{cases} r^7 = 1 \\ 7\alpha = 0 + k\cdot 2\pi \end{cases} \Rightarrow \begin{cases} r = 1 \\ \alpha = k\cdot\frac{2\pi}{7} \end{cases} \quad k = 0, 1, \dots, 6$$

$w_0 = cis(0) = 1$, $w_1 = cis\left(\frac{2\pi}{7}\right)$, $w_2 = cis\left(\frac{4\pi}{7}\right)$, $w_3 = cis\left(\frac{6\pi}{7}\right)$

$w_4 = cis\left(\frac{8\pi}{7}\right)$, $w_5 = cis\left(\frac{10\pi}{7}\right)$, $w_6 = cis\left(\frac{12\pi}{7}\right)$

b) $z^5 = -32 = 32\,cis(\pi)$



on pose $z = t\,cis(\alpha)$

$$\Rightarrow z^5 = t^5\,cis(5\alpha) = 32\,cis(\pi) \Rightarrow \begin{cases} t^5 = 32 \\ 5\alpha = \pi + k\cdot 2\pi \end{cases} \Leftrightarrow \begin{cases} t = 2 \\ \alpha = \frac{\pi}{5} + k\cdot\frac{2\pi}{5} \end{cases},$$

$k = 0, 1, \dots, 4$

$z_0 = 2\,cis\left(\frac{\pi}{5}\right)$

$z_1 = 2\,cis\left(\frac{\pi}{5} + \frac{2\pi}{5}\right) = 2\,cis\left(\frac{3\pi}{5}\right)$

$z_2 = 2\,cis\left(\frac{\pi}{5} + 2\cdot\frac{2\pi}{5}\right) = 2\,cis(\pi)$

$z_3 = 2\,cis\left(\frac{\pi}{5} + 3\cdot\frac{2\pi}{5}\right) = 2\,cis\left(\frac{7\pi}{5}\right)$

$z_4 = 2\,cis\left(\frac{\pi}{5} + 4\cdot\frac{2\pi}{5}\right) = 2\,cis\left(\frac{9\pi}{5}\right)$

Ex 1.2.10

a) $(a+bi)^2 = 1 \iff z_{1,2} = \pm 1$

b) $(a+bi)^2 = i \iff a^2 - b^2 + 2abi = i \iff \begin{cases} a^2+b^2 = 1 \\ a^2-b^2 = 0 \\ 2ab = 1 \end{cases} \begin{vmatrix} 1 \\ 1 \end{vmatrix} \begin{vmatrix} 1 \\ -1 \end{vmatrix}$ (module de z)

$\iff \begin{cases} 2a^2 = 1 \\ 2b^2 = 1 \\ ab = 1/2 \end{cases} \iff \begin{cases} a = \pm\sqrt{\frac{1}{2}} = \pm \frac{\sqrt{2}}{2} \\ b = \pm \frac{\sqrt{2}}{2} \end{cases}$

$\Rightarrow z_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$ et $z_2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

c) $(a+bi)^2 = -i \iff \begin{cases} a^2+b^2 = 1 \\ a^2-b^2 = 0 \\ 2ab = -1 \end{cases} \begin{vmatrix} 1 \\ 1 \end{vmatrix} \begin{vmatrix} 1 \\ -1 \end{vmatrix} \iff \begin{cases} 2a^2 = 1 \\ 2b^2 = 1 \\ ab = -1/2 \end{cases} \iff \begin{cases} a = \pm\frac{\sqrt{2}}{2} \\ b = \mp \frac{\sqrt{2}}{2} \end{cases}$

$\Rightarrow z_1 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$ et $z_2 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

d) $(a+bi)^2 = -9 \iff z_{1,2} = \pm\sqrt{-9} = \pm\sqrt{9\cdot(-1)} = \pm 3\sqrt{-1} = \pm 3i$

e) $(a+bi)^2 = 3+4i \iff \begin{cases} a^2+b^2 = \sqrt{9+16} = 5 \\ a^2-b^2 = 3 \\ 2ab = 4 \end{cases} \begin{vmatrix} 1 \\ 1 \end{vmatrix} \begin{vmatrix} 1 \\ -1 \end{vmatrix} \iff \begin{cases} 2a^2 = 8 \\ 2b^2 = 2 \\ ab = 2 \end{cases}$

$\iff \begin{cases} a = \pm 2 \\ b = \pm 1 \end{cases} \Rightarrow z_1 = 2+i$ et $z_2 = -2-i$

f) $(a+bi)^2 = -5+12i \iff \begin{cases} a^2+b^2 = 13 \\ a^2-b^2 = -5 \\ 2ab = 12 \end{cases} \begin{vmatrix} 1 \\ 1 \end{vmatrix} \begin{vmatrix} 1 \\ -1 \end{vmatrix} \iff \begin{cases} 2a^2 = 8 \\ 2b^2 = 18 \\ ab = 6 \end{cases}$

$\iff \begin{cases} a = \pm 2 \\ b = \pm 3 \end{cases} \Rightarrow z_1 = 2+3i$ et $z_2 = -2-3i$