Ex 2.9.1

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

a) $f(x) = 4$

Rem : fct constante $\Rightarrow$ pente de la tgte en tout point $= 0$

$$f'(a) = \lim_{x \to a} \frac{4-4}{x-a} = \lim_{x \to a} 0 = 0 \Rightarrow f'(x) = 0$$

b) $f(x) = 2x - 5$ Rem : fct affine $\Rightarrow$ pente de la tgte en tout point $= m = 2$

$$f'(a) = \lim_{x \to a} \frac{2x-5-(2a-5)}{x-a} = \lim_{x \to a} \frac{2x-2a}{x-a} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to a} \frac{2\cancel{(x-a)}}{\cancel{x-a}} = 2 \Rightarrow f'(x) = 2$$

c) $f(x) = x^2 - 1$

$$f'(a) = \lim_{x \to a} \frac{x^2-1-(a^2-1)}{x-a} = \lim_{x \to a} \frac{x^2-a^2}{x-a} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to a} \frac{(x+a)\cancel{(x-a)}}{\cancel{x-a}} = \lim_{x \to a} (x+a) = 2a$$

$$\Rightarrow f'(x) = 2x$$

d) $f(x) = \dfrac{1}{3x+1}$

$$f'(a) = \lim_{x \to a} \frac{\frac{1}{3x+1} - \frac{1}{3a+1}}{x-a} = \lim_{x \to a} \frac{3a+1-(3x+1)}{(3x+1)(3a+1)} \cdot \frac{1}{x-a}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to a} \frac{3a-3x}{(3x+1)(3a+1)(x-a)} = \lim_{x \to a} \frac{-3\cancel{(x-a)}}{(3x+1)(3a+1)\cancel{(x-a)}} = \frac{-3}{(3a+1)^2}$$

$$\Rightarrow f'(x) = \frac{-3}{(3x+1)^2}$$

e) $f(x) = \dfrac{4x-1}{x+1}$

$$f'(a) = \lim_{x \to a} \frac{\frac{(4x-1)}{x+1} - \frac{(4a-1)}{a+1}}{x-a} = \lim_{x \to a} \frac{\overbrace{(4x-1)(a+1) - (4a-1)(x+1)}^{4ax+4x-a-1-(4ax-x+4a-1)}}{(x+1)(a+1)} \cdot \frac{1}{x-a}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to a} \frac{5x-5a}{(x+1)(a+1)(x-a)} = \lim_{x \to a} \frac{5\cancel{(x-a)}}{(x+1)(a+1)\cancel{(x-a)}} \lim_{x \to a} \frac{5}{(x+1)(a+1)}$$

$$= \frac{5}{(a+1)^2} \qquad \Rightarrow f'(x) = \frac{5}{(x+1)^2}$$

e) $f(x) = \sqrt{x-3}$

$$f'(a) = \lim_{x \to a} \frac{\sqrt{x-3} - \sqrt{a-3}}{x-a} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to a} \frac{\sqrt{x-3} - \sqrt{a-3}}{x-a} \cdot \frac{\sqrt{x-3} + \sqrt{a-3}}{\sqrt{x-3} + \sqrt{a-3}}$$

$$= \lim_{x \to a} \frac{x-3-(a-3)}{(x-a)(\sqrt{x-3}+\sqrt{a-3})} = \lim_{x \to a} \frac{\cancel{x-a}}{\cancel{(x-a)}(\sqrt{x-3}+\sqrt{a-3})} = \frac{1}{2\sqrt{a-3}}$$

$$\Rightarrow f'(x) = \frac{1}{2\sqrt{a-3}}$$

Ex 2.9.2

$f(x) = -x^2 + x + 2$

a) $$f'(a) = \lim_{x \to a} \frac{-x^2+x+2-(-a^2+a+2)}{x-a} = \lim_{x \to a} \frac{-x^2+x+a^2-a}{x-a}$$

$$= \lim_{x \to a} \frac{-(x^2-a^2)+x-a}{x-a} = \lim_{x \to a} \frac{-(x+a)(x-a)+(x-a)}{x-a}$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}(-(x+a)+1)}{\cancel{x-a}} = -2a+1 \qquad \Rightarrow \quad f'(x) = -2x+1$$

b) • coupe l'axe Oy : ord. à l'O. : $f(0) = 2 \quad \Rightarrow \quad A(0;2)$

$\Rightarrow$ pente : $m_A = f'(0) = -2 \cdot 0 + 1 = 1$

• coupe l'axe Ox : zéros de f : $f(x) = 0 \iff -x^2+x+2 = 0$

$$\iff -(x^2-x-2) = 0$$

$$\iff -(x-2)(x+1) = 0$$

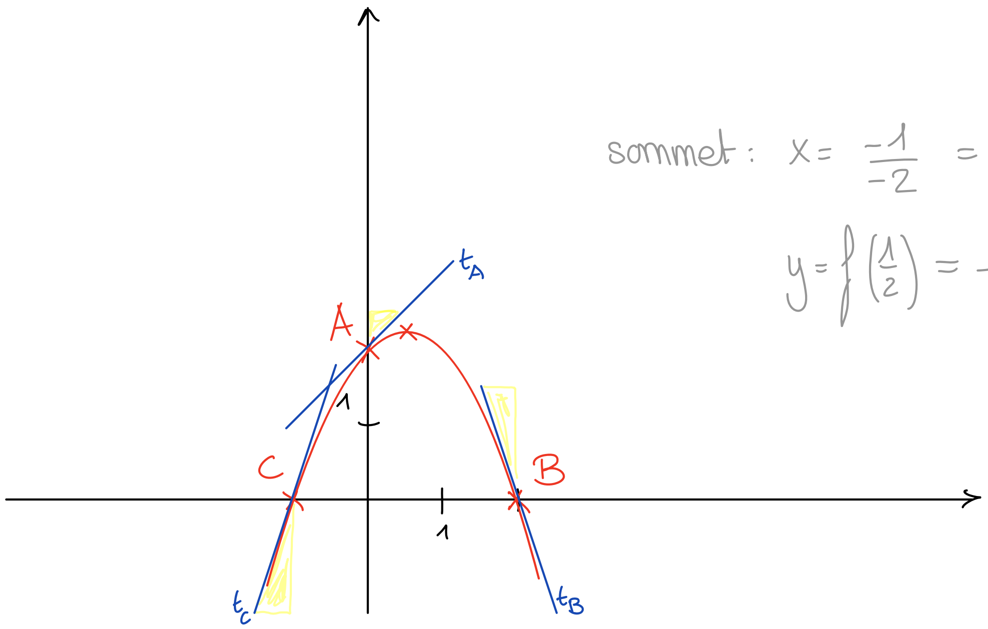
$$\quad\;\; 2 \qquad\; -1$$

$\Rightarrow B(2;0)$ et $C(-1;0)$

$\Rightarrow$ pente : $m_B = f'(2) = -2 \cdot 2 + 1 = -3$

$m_C = f'(-1) = -2 \cdot (-1) + 1 = 3$

c)



sommet : $x = \frac{-1}{-2} = \frac{1}{2}$

$y = f\left(\frac{1}{2}\right) = -\frac{1}{4} + \frac{1}{2} + 2 = \frac{9}{4}$

Ex 2.9.7

a) $f(x) = 47$ $\quad f'(x) = 0$

b) $f(x) = 3x$ $\quad f'(x) = 3$

c) $f(x) = x^5$ $\quad f'(x) = 5x^4$

d) $f(x) = 8x^7$ $\quad f'(x) = 56x^6$

e) $f(x) = 5x^0 = 5 \Rightarrow f'(x) = 0$

f) $f(x) = \frac{1}{3}x^3$ $\quad f'(x) = \frac{1}{3} \cdot 3x^2 = x^2$

g) $f(x) = x^3 + x^2 + x + 1$ $\quad f'(x) = 3x^2 + 2x + 1$

h) $f(x) = 7x^4 - 3x + 8$ $\quad f'(x) = 28x^3 - 3$

i) $f(x) = x^2 + 5x - 6$ $\quad f'(x) = 2x + 5$

j) $f(x) = x^3 + 5x^2 - 2x + 4$ $\quad f'(x) = 3x^2 + 10x - 2$

k) $f(x) = \frac{2}{3}x^3 - \frac{5}{2}x^2 + 2x + 4$

$f'(x) = \frac{2}{3} \cdot 3x^2 - \frac{5}{2} \cdot 2x + 2 = 2x^2 - 5x + 2$

l) $f(x) = 2x^5 - \frac{7}{6}x^3 + \frac{3}{4}x^2 - x + \sqrt{2}$

$f'(x) = 10x^4 - \frac{7}{6} \cdot 3x^2 + \frac{3}{4} \cdot 2x - 1 + 0$

$= 10x^4 - \frac{7}{2}x^2 + \frac{3}{2}x - 1$

Ex 2.9.8

a) $\left((x+1)(x-3)\right)' = $

$1\cdot(x-3) + 1\cdot(x+1) = x-3+x+1 = 2x-2$

$(x^2-3x+x-3)' = (x^2-2x-3)' = \underline{2x-2}$

$u = x+1 \qquad v = x-3$
$u' = 1 \qquad v' = 1$

b) $\left(x(x^2+5)\right)' = $

$1\cdot(x^2+5) + x\cdot 2x = x^2+5+2x^2 = 3x^2+5$

$(x^3+5x)' = 3x^2+5$

$u = x \qquad v = x^2+5$
$u' = 1 \qquad v' = 2x$

c) $\left((7x^2-4x+3)(5-2x)\right)' = (14x-4)(5-2x) + (7x^2-4x+3)\cdot(-2)$

$u = 7x^2-4x+3 \qquad v = 5-2x$
$u' = 14x-4 \qquad v' = -2$

$= 70x - 28x^2 - 20 + 8x - 14x^2 + 8x - 6$

$= \underline{-42x^2 + 86x - 26}$

d) $\left((2x-1)(2-2x)(1+x)\right) = 2(2-2x)(1+x) + (2x-1)\cdot(-2)(1+x) + (2x-1)(2-2x)\cdot 1$

$= (4-4x)(1+x) + (2x-1)(-2-2x) + 4x-4x^2-2+2x$

$= 4+4x-4x-4x^2-4x-4x^2+2+2x+4x-4x^2-2+2x$

$= \underline{-12x^2+4x+4}$

e) $\left(\dfrac{4-3x}{2x-1}\right)' = \dfrac{-3(2x-1)-(4-3x)\cdot 2}{(2x-1)^2} = \dfrac{-6x+3-8+6x}{(2x-1)^2} = \underline{\dfrac{-5}{(2x-1)^2}}$

$u = 4-3x \qquad v = 2x-1$
$u' = -3 \qquad v' = 2$

f) $\left(\dfrac{x-2}{3-x}\right)' = \dfrac{1(3-x)-(x-2)\cdot(-1)}{(3-x)^2} = \dfrac{3-x+x-2}{(3-x)^2} = \underline{\dfrac{1}{(3-x)^2}}$

g) $\left(\dfrac{5}{2x^2-1}\right)' = 5\cdot\left(\dfrac{1}{2x^2-1}\right)' = 5\cdot\dfrac{-4x}{(2x^2-1)^2} = \underline{-\dfrac{20x}{(2x^2-1)^2}}$

$v = 2x^2-1$
$v' = 4x$

h) $\left(\dfrac{x^3-10x^2}{1-x}\right)' = \dfrac{(3x^2-20x)(1-x) - (x^3-10x^2)(-1)}{(1-x)^2}$

$u = x^3 - 10x^2$
$u' = 3x^2 - 20x$

$v = 1-x$
$v' = -1$

$= \dfrac{3x^2 - 3x^3 - 20x + 20x^2 + x^3 - 10x^2}{(1-x)^2}$

$= \dfrac{-2x^3 + 13x^2 - 20x}{(1-x)^2}$

i) $\left(\dfrac{8x^2-8x+3}{4x^2-1}\right)'$

$u = 8x^2-8x+3$
$u' = 16x-8$

$v = 4x^2-1$
$v' = 8x$

$= \dfrac{(16x-8)(4x^2-1) - (8x^2-8x+3)\cdot 8x}{(4x^2-1)^2}$

$= \dfrac{64x^3 - 16x - 32x^2 + 8 - 64x^3 + 64x^2 - 24x}{(4x^2-1)^2} = \dfrac{32x^2 - 40x + 8}{(4x^2-1)^2}$

j) $\left(\dfrac{x^3}{x+1}\right)' = \dfrac{3x^2(x+1) - x^3\cdot 1}{(x+1)^2} = \dfrac{3x^3 + 3x^2 - x^3}{(x+1)^2} = \dfrac{2x^3+3x^2}{(x+1)^2}$

k) $\left(1 + \dfrac{1}{x} - \dfrac{2}{x^2}\right)' = 0 - \dfrac{1}{x^2} - 2\cdot\left(-\dfrac{2x}{x^4}\right) = -\dfrac{1}{x^2} + \dfrac{4}{x^3} = \dfrac{-x+4}{x^3}$

l) $\left(\dfrac{x^3-4}{3x} + x\right)' = \dfrac{3x^2\cdot 3x - (x^3-4)\cdot 3}{9x^2} + 1 = \dfrac{9x^3 - 3x^3 + 12}{9x^2} + 1$

$= \dfrac{6x^3+12}{9x^2} + \dfrac{9x^2}{9x^2} = \dfrac{\overbrace{6x^3 + 9x^2 + 12}^{3(2x^3+3x^2+4)}}{9x^2} = \dfrac{2x^3+3x^2+4}{3x^2}$

Ex 2.9.9

a) $f(x) = mx + h \implies f'(x) = m$

b) $f(x) = (w-1)x^3 + w(x-3) \implies f'(x) = 3(w-1)x^2 + w$

c) $f(x) = ax^2 + bx + c \implies f'(x) = 2ax + b$

d) $f(x) = \dfrac{ax+b}{cx+d}$

$$\implies f'(x) = \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}$$

e) $f(x) = \dfrac{x}{x+t}$

$$\implies f'(x) = \frac{1(x+t) - 1x}{(x+t)^2} = \frac{t}{(x+t)^2}$$

f) $f(x) = \dfrac{3x^2 + 2ax + 2a}{x^2 + ax + a}$

$$\implies f'(x) = \frac{(6x+2a)(x^2+ax+a) - (3x^2+2ax+2a)(2x+a)}{(x^2+ax+a)^2}$$

$$= \frac{6x^3 + 6ax^2 + 6ax + 2ax^2 + 2a^2x + 2a^2 - (6x^3 + 3ax^2 + 4ax^2 + 2a^2x + 4ax + 2a^2)}{(x^2+ax+a)^2}$$

$$= \frac{ax^2 + 2ax}{(x^2+ax+a)^2} = \frac{ax(x+2)}{(x^2+ax+a)^2}$$

Ex 2.9.10

Rappel : $\boxed{(u^n)' = n \cdot u^{n-1} \cdot \underbrace{u'}_{\text{dérivée interne}}}$

a) $f(x) = (2x+3)^4$

$f'(x) = 4(2x+3)^3 \cdot \underbrace{2}_{\text{dérivée interne}} = \underline{8(2x+3)^3}$

b) $f(x) = (3-x)^5$

$f'(x) = 5(3-x)^4 \cdot \underbrace{(-1)}_{\text{dérivée interne}} = \underline{-5(3-x)^4}$

c) $f(x) = (x^2+5x+1)^3$

$f'(x) = \underline{3(x^2+5x+1)^2 \underbrace{(2x+5)}_{\text{dérivée interne}}}$

d) $f(x) = (x^3-2x)^7$

$f'(x) = \underline{7(x^3-2x)^6 \underbrace{(3x^2-2)}_{\text{dérivée interne}}}$

Rappel : $\boxed{(u \cdot v)' = u'v + uv'}$

e) $f(x) = x^2(5x+2)^3$

$u = x^2 \qquad v = (5x+2)^3$

$u' = 2x \qquad v' = 3(5x+2)^2 \cdot \underline{5} = 15(5x+2)^2$

$f'(x) = 2x(5x+2)^3 + 15x^2(5x+2)^2$ (mise en évidence)

$= x(5x+2)^2\left[\overbrace{2(5x+2)}^{10x+4} + 15x\right]$

$= \underline{x(5x+2)^2(25x+4)}$

f) $f(x) = (2+x)^2(1-x)^3$

$u = (2+x)^2$ $\quad$ $v = (1-x)^3$

$u' = 2(2+x)\cdot 1$ $\quad$ $v' = 3(1-x)^2\cdot(-1)$

$= 2(2+x)$ $\quad$ $= -3(1-x)^2$

$f'(x) = 2(2+x)(1-x)^3 - 3(2+x)^2(1-x)^2$ (mee)

$= (2+x)(1-x)^2\left[2(1-x) - 3(2+x)\right]$

$= (2+x)(1-x)^2(2-2x-6-3x)$

$= \underline{(2+x)(1-x)^2(-5x-4)} = \underline{-(2+x)(1-x)^2(5x+4)}$

g) $f(x) = (2x+5)^3(3x-1)^4$

$u = (2x+5)^3$ $\quad$ $v = (3x-1)^4$

$u' = 3(2x+5)^2\cdot 2$ $\quad$ $v' = 4(3x-1)^3\cdot 3$

$= 6(2x+5)^2$ $\quad$ $= 12(3x-1)^3$

$f'(x) = 6(2x+5)^2(3x-1)^4 + 12(2x+5)^3(3x-1)^3$ (mee)

$= 6(2x+5)^2(3x-1)^3\left[(3x-1) + \overbrace{2(2x+5)}^{4x+10}\right]$

$= \underline{6(2x+5)^2(3x-1)^3(7x+9)}$

h) $f(x) = (1-3x)^2(2-x)(x+3)^3$

$$f'(x) = -6(1-3x)(2-x)(x+3)^3 - (1-3x)^2(x+3)^3 + 3(1-3x)^2(2-x)(x+3)^2$$
$$= (1-3x)(x+3)^2[-6(2-x)(x+3) - (1-3x)(x+3) + 3(1-3x)(2-x)]$$
$$= (1-3x)(x+3)^2(6x^2+6x-36+3x^2+8x-3+9x^2-21x+6)$$
$$= (1-3x)(x+3)^2(18x^2-7x-33)$$

Rappel : $\boxed{\left(\frac{1}{v}\right)' = -\frac{v'}{v^2}}$

i) $f(x) = \frac{1}{(x^2+3)^2}$

$v = (x^2+3)^2$

$v' = 2(x^2+3)\cdot 2x = 4x(x^2+3)$

$$f'(x) = -\frac{4x(x^2+3)}{(x^2+3)^4} = -\frac{4x}{(x^2+3)^3}$$

Rappel : $\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}$

j) $f(x) = \frac{x}{(3x+2)^2}$

$u = x$ $\quad\quad$ $v = (3x+2)^2$

$u' = 1$ $\quad\quad$ $v' = 2(3x+2)\cdot 3 = 6(3x+2)$

$$f'(x) = \frac{(3x+2)^2 - 6x(3x+2)}{(3x+2)^4}$$ (mee)

$$= \frac{\cancel{(3x+2)}\left[(3x+2) - 6x\right]}{(3x+2)^{\cancel{4}\,3}} = \frac{-3x+2}{(3x+2)^3}$$

k) $f(x) = \frac{(1-x)^3}{(1+x)^2}$

$u = (1-x)^3$ $\quad\quad$ $v = (1+x)^2$

$u' = 3(1-x)^2\cdot(-1)$ $\quad\quad$ $v' = 2(1+x)$

$\quad = -3(1-x)^2$

$$f'(x) = \frac{-3(1-x)^2(1+x)^2 - 2(1-x)^3(1+x)}{(1+x)^4}$$

$$= \frac{-(1-x)^2\cancel{(1+x)}\left[\overbrace{3(1+x) + 2(1-x)}^{3+3x+2-2x}\right]}{(1+x)^{\cancel{4}\,3}} = -\frac{(1-x)^2(x+5)}{(1+x)^3}$$

l) $f(x) = \dfrac{x(x-3)^2}{(x-2)^2}$

$u = x(x-3)^2$ $\qquad$ $v = (x-2)^2$

$u' = 1\cdot(x-3)^2 + x\cdot 2(x-3)\cdot 1$ $\qquad$ $v' = 2(x-2)$

$= (x-3)^2 + 2x(x-3)$

$= (x-3)[(x-3)+2x]$

$= (x-3)(3x-3)$

$= 3(x-3)(x-1)$

$$f'(x) = \frac{3(x-3)(x-1)(x-2)^2 - 2x(x-3)^2(x-2)}{(x-2)^4}$$

$$= \frac{(x-3)\cancel{(x-2)}\,[3\underbrace{(x-1)(x-2)}_{x^2-3x+2} - 2x(x-3)]}{(x-2)^{\cancel{4}\,3}}$$

$$= \frac{(x-3)(3x^2-9x+6-2x^2+6x)}{(x-2)^3} = \frac{(x-3)(\overbrace{x^2-3x+6}^{\Delta = 9-24<0})}{(x-2)^3}$$

Ex 2.9.11

a) $f(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3} = \dfrac{1}{3\sqrt[3]{x^2}}$

b) $f(x) = \sqrt[5]{x} = x^{1/5} \Rightarrow f'(x) = \frac{1}{5}x^{-4/5} = \dfrac{1}{5\sqrt[5]{x^4}}$

c) $f(x) = \sqrt[7]{x^4} = x^{4/7} \Rightarrow f'(x) = \frac{4}{7}x^{-3/7} = \dfrac{4}{7\sqrt[7]{x^3}}$

d) $f(x) = \sqrt{8x^2-5x+3}$

$\Rightarrow f'(x) = \dfrac{16x-5}{2\sqrt{8x^2-5x+3}}$ $\qquad$ en utilisant directement $(\sqrt{u})' = \dfrac{u'}{2\sqrt{u}}$

e) $f(x) = \sqrt{x^2+4} \implies f'(x) = \dfrac{2x}{2\sqrt{x^2+4}} = \dfrac{x}{\sqrt{x^2+4}}$

f) $f(x) = \sqrt{(4x^2-2x)^3} = (4x^2-2x)^{3/2}$

$$f'(x) = \frac{3}{2}(4x^2-2x)^{1/2}\cdot(8x-2) = 3\sqrt{4x^2-2x}\,(4x-1)$$

g) $f(x) = \sqrt[3]{x^2+x+1} = (x^2+x+1)^{1/3}$

$$f'(x) = \frac{1}{3}(x^2+x+1)^{-2/3}(2x+1) = \frac{2x+1}{3\sqrt[3]{(x^2+x+1)^2}}$$

h) $f(x) = \sqrt[3]{(1-x^2)^2} = (1-x^2)^{2/3}$

$$f'(x) = \frac{2}{3}(1-x^2)^{-1/3}\cdot(-2x) = \frac{-4x}{3\sqrt[3]{1-x^2}}$$

i) $f(x) = \dfrac{1}{\sqrt{x}}$

avec $\left(\dfrac{1}{v}\right)' = -\dfrac{v'}{v^2}$

$$f'(x) = -\frac{\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} = -\frac{\frac{1}{2\sqrt{x}}}{x} = -\frac{1}{2\sqrt{x}}\cdot\frac{1}{x} = -\frac{1}{2x\sqrt{x}}$$

Variante : avec $(x^n)' = nx^{n-1}$

$$f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$$

$$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2}\cdot\frac{1}{\sqrt{x^3}} = -\frac{1}{2\sqrt{x^3}}$$

j) $f(x) = \dfrac{1}{\sqrt[3]{x^2}} = x^{-2/3}$

$$f'(x) = -\frac{2}{3}x^{-5/3} = -\frac{2}{3\sqrt[3]{x^5}}$$

k) $f(x) = (1+x)\sqrt{1-x}$

$u = 1+x \qquad v = \sqrt{1-x}$

$u' = 1 \qquad v' = \dfrac{-1}{2\sqrt{1-x}}$

$$f'(x) = \sqrt{1-x} + (1+x)\cdot\frac{-1}{2\sqrt{1-x}} = \sqrt{1-x} - \frac{1+x}{2\sqrt{1-x}}$$

$$= \frac{\sqrt{1-x}\cdot 2\sqrt{1-x}}{2\sqrt{1-x}} - \frac{1+x}{2\sqrt{1-x}}$$

$$= \frac{2(1-x)-(1+x)}{2\sqrt{1-x}} = \frac{2-2x-1-x}{2\sqrt{1-x}} = \frac{1-3x}{2\sqrt{1-x}}$$

l) $f(x) = \sqrt{\dfrac{3x-2}{x+1}}$

$u = \dfrac{3x-2}{x+1} \qquad u' = \dfrac{3(x+1)-(3x-2)\cdot 1}{(x+1)^2} = \dfrac{3x+3-3x+2}{(x+1)^2} = \dfrac{5}{(x+1)^2}$

$$f'(x) = \frac{\frac{5}{(x+1)^2}}{2\sqrt{\frac{3x-2}{x+1}}} = \frac{5}{(x+1)^2}\cdot\frac{1}{2\sqrt{\frac{3x-2}{x+1}}} = \frac{5}{2(x+1)^2}\sqrt{\frac{x+1}{3x-2}}$$